Analysis of Surface Heating by Induction

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An analytical solution of the transient temperature distribution of an electrically conducting material exposed to electromagnetic radiation is obtained. The solid material is assumed to be semi-infinite with a specified initial uniform ambient temperature. The surface is assumed to be subjected to convective heat losses. However, adiabatic surfaces boundary condition and specified surface temperature boundary condition are also treated as special cases. The effect of the applied current frequency is discussed, and the temperature rise during high-frequency electromagnetic heating, which is of particular importance in surface modification of materials, is investigated.

Keywords
analysis, frequency, induction heating, penetration depth, steel

1. Introduction

HEATING by means of electromagnetic waves has wide application in industry—for example, microwave ovens, and induction furnaces for melting and surface modification purposes. Induction heating systems offer a number of advantages over furnaces. Energy savings are provided through fast heating and high production rates. Other advantages include ease of automation and control, low maintenance requirements, and quiet, safe, and clean working conditions (Ref 1).

Numerous analytical and experimental studies of heat transfer through electromagnetic heating have been conducted in recent years. When electromagnetic energy is sent through the exposed surface, depending on the frequency applied, a spatially varying heat generation occurs inside the electrically conducting material. If the surface is exposed to an ambient atmosphere and convection and/or radiation occurs, then it is most likely that the maximum temperature lies inside the material and not on the surface. Therefore, the initiation of phase changes and the volume expansion that is due to the phase change occur inside the material, rather than at the surface. This affects the surface characteristics during the surface treatment process and may also cause residual stresses to build up and microcracks to develop.

One-dimensional transient heat conduction analysis in a slab for induction heating is described by Sahin et al. (Ref 2). In their solution, the slab material was assumed to be insulated and no steady-state solution was obtained. However, in practice, insulation may not always be possible. Then, the Biot number (Bi) plays an important role during the heating process as a controlling parameter. The present study provides an analytical solution of the transient temperature distribution of an electrically conducting material exposed to electromagnetic radiation. The solid material is assumed to be semi-infinite with a specified initial constant ambient temperature. The surface is assumed to be subjected to convective heat losses. However, an insulated surface boundary condition and a specified surface temperature boundary condition are also treated as special cases.

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2. Problem Statement

A semi-infinite electrically conducting material is to be heated from the surface by induction heating. The governing differential equation for the current density, J (A/m^2), in this conducting material is derived from Maxwell's equations and is (Ref 3):

$$\nabla^2 J = \frac{\mu}{\rho}\frac{\partial J}{\partial t} \quad \text{(Eq 1)}$$

where μ is magnetic permeability and ρ is electrical resistivity. It should be noted that $\mu = \mu_0\mu^r$, where $\mu_0 = 4\pi 10^{-7}$ and μ_r is the relative permeability, which is a function of the magnetic intensity (Ref 3). In a one-dimensional case and for a sinusoidally varying current in the form:

Symbols

a	thermal diffusivity (m^2/s)
Bi	$h\delta/2k$, dimensionless Biot number
f	$\omega/2\pi$, frequency (Hz)
h	heat-transfer coefficient (W/m$^2\cdot$K)
J	current density, A/m^2
k	thermal conductivity (W/mK)
P	surface power density (W/m^2)
q''	dimensionless local heat flux
$\dot{q}$	heat generation rate (W/m^3)
t	time (s)
T	temperature, K
T_∞	ambient temperature (K)
x	position in the material (m)
δ	depth of penetration (m)
μ	magnetic permeability (Ωs/m)
μ_0	reference magnetic permeability (= $4\pi 10^{-7}$ Ωs/m)
μ_r	μ/μ_0, relative permeability
ρ	electrical resistivity (Ωm)
ω	angular frequency (1/s)
θ	$(T - T_\infty)/(\delta P/2k)$, dimensionless temperature
τ	$4at/\delta^2$, dimensionless time
ξ	$2x/\delta$, dimensionless coordinate

$$J(x, t) = J(x)\cos(\omega t) = \mathrm{Re}[J(x)e^{i\omega t}] \quad \text{(Eq 2)}$$

where x is the distance into the material from the surface and $\mathrm{Re}[\eta]$ represents the real part of a function η, Eq 1 is reduced to:

$$\frac{d^2J}{dx^2} = \frac{i\omega\mu}{\rho}J \quad \text{(Eq 3)}$$

The general solution of Eq 3 is given by:

$$J(x) = C_1 \exp\left(\sqrt{\frac{i\omega\mu}{\rho}}\,x\right) + C_2 \exp\left(-\sqrt{\frac{i\omega\mu}{\rho}}\,x\right) \quad (\mathrm{A/m^2}) \quad \text{(Eq 4)}$$

The boundary conditions are:

$$J(0) = J_s \quad \text{(Eq 5)}$$

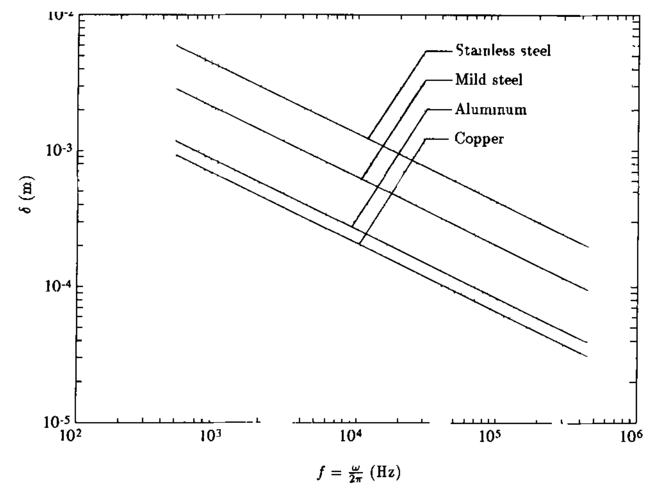

Fig. 1 Penetration depth as a function of frequency at 20 °C and $\mu_r = 10$ for several materials

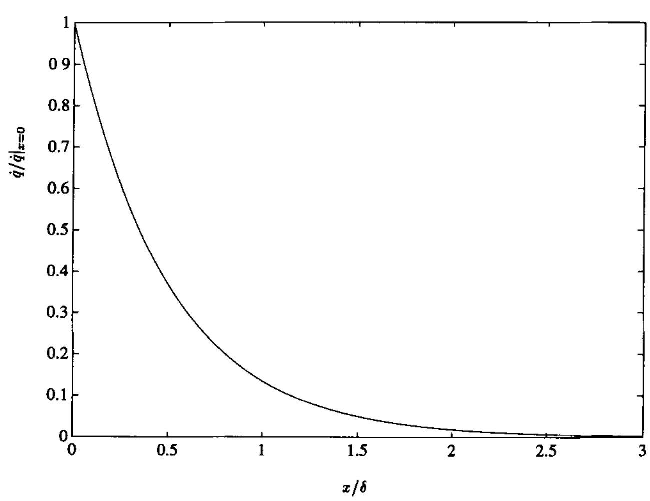

Fig. 2 Energy generation fraction inside the material due to induction heating as a function of distance from the surface

on the surface and:

$$J(\infty) = \text{finite} \quad \text{(Eq 6)}$$

where J_s is the value of the current density at the surface.

The solution of Eq 3 that satisfies the boundary conditions (Eq 5 and 6) is:

$$J(x) = J_s \exp\left(-\sqrt{\frac{i\omega\mu}{\rho}}\,x\right) = J_s \exp\left(-\sqrt{2i}\,\frac{x}{\delta}\right) \quad \text{(Eq 7)}$$

where

$$\delta = \sqrt{\frac{2\rho}{\mu\omega}} \quad \text{(Eq 8)}$$

is the penetration depth in which 86.5% of the power consumption takes place (Ref 3). It should be noted that an increase in the angular frequency, ω, decreases the penetration depth, δ. This is a very important parameter in surface treatment processes. Figure 1 shows the variation of penetration depth, δ, with frequency, $f = \omega/2\pi$ (hertz), for different materials at an ambient temperature of $T_\infty = 20$ °C and relative permeability of $\mu_r = \mu/\mu_0 = 10$. The electrical resistivity values at 20 °C for the materials shown in Fig. 1 are 0.695, 0.16, 0.027, and 0.017 μΩm for stainless steel (19.11% Cr, 8.14% Ni, and 0.6% W), mild steel (0.23% C), aluminum, and copper, respectively.

In the range of commercial frequencies (50 to 450 kHz), the penetration depth of mild steel (0.23% C) varies from 3 mm to 100 μm at 20 °C. The penetration depth can be less than 100 μm for copper and aluminum if the frequency is increased above 50 kHz.

The heat generation due to the current density given by Eq 7 is:

$$\dot{q} = \rho|J(x)|^2 = 2\frac{P}{\delta}\exp\left(-2\,\frac{x}{\delta}\right) \quad (\mathrm{W/m^3}) \quad \text{(Eq 9)}$$

where P is the surface power density or, in other words, the total power consumption per unit surface area, which is given by:

$$P = \int_0^\infty \dot{q}\,dx \quad (\mathrm{W/m^2})$$

The generation of heat inside the material due to induction heating is plotted against the depth from the surface in Fig. 2. It is clear that within one penetration depth ($x = \delta$), the heat generation rate drops by 86.5% of its value at the surface. In general, the heat generation is practically negligible beyond two penetration depths ($x > 2\delta$).

The governing partial-differential equation for the transient one-dimensional temperature distribution in the semi-infinite domain with heat generation has the form:

$$\frac{1}{a}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{\dot{q}}{k} \qquad \text{(Eq 10)}$$

where a is the thermal diffusivity, k is the thermal conductivity, and $\dot{q}$ is the heat generation given by Eq 9.

The material is assumed to be initially at the uniform ambient temperature:

$$T(x, 0) = T_\infty \qquad \text{(Eq 11)}$$

The boundary conditions are:

$$-k\frac{\partial T(0, t)}{\partial x}=h[T_\infty - T(0, t)] \qquad \text{(Eq 12)}$$

and

$$T(\infty, t) = \text{finite} \qquad \text{(Eq 13)}$$

Note that $h = 0$ and $h \to \infty$ in Eq 12 correspond to the adiabatic surface and the constant surface temperature cases, respectively.

3. Solution for Temperature Profile

Following the procedure given by Sahin (Ref 4), the solution for the problem presented is:

$$\begin{aligned}T(x, t) = T_\infty + \frac{\delta P}{2k}\Bigg\{&-\exp\left(-2\frac{x}{\delta}\right)+\frac{1}{2}\exp\left(\frac{4at}{\delta^2}+2\frac{x}{\delta}\right)\\ &\cdot \operatorname{erfc}\left(\frac{2\sqrt{at}}{\delta}+\frac{x}{2\sqrt{at}}\right)+\frac{1}{2}\exp\left(\frac{4at}{\delta^2}-2\frac{x}{\delta}\right)\\ &\cdot \operatorname{erfc}\left(\frac{2\sqrt{at}}{\delta}-\frac{x}{2\sqrt{at}}\right)-\left(1+\frac{2k}{\delta h}\right)\exp\left[\left(\frac{h}{k}\right)^2 at+\frac{h}{k}x\right]\\ &\cdot \operatorname{erfc}\left(\frac{h}{k}\sqrt{at}+\frac{x}{2\sqrt{at}}\right)+\left(1+\frac{2k}{\delta h}\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right)\\ &+\left(\frac{\delta h}{\delta h-2k}\right)\exp\left[\left(\frac{h}{k}\right)^2 at+\frac{h}{k}x\right]\operatorname{erfc}\left(\frac{h}{k}\sqrt{at}+\frac{x}{2\sqrt{at}}\right)\\ &-\left(\frac{\delta h}{\delta h-2k}\right)\exp\left(\frac{4at}{\delta^2}+2\frac{x}{\delta}\right)\operatorname{erfc}\left(\frac{2\sqrt{at}}{\delta}+\frac{x}{2\sqrt{at}}\right)\Bigg\}\end{aligned} \qquad \text{(Eq 14)}$$

where erfc (η) is the complementary error function given by:

$$\operatorname{erfc}(\eta) = 1 - \operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_\eta^\infty e^{-\zeta^2}d\zeta$$

Equation 14 can be nondimensionalized by using the following dimensionless parameters:

$$\theta = \frac{T - T_\infty}{(\delta P/2k)} \qquad \text{(Eq 15)}$$

$$\xi = \frac{x}{\delta/2} \qquad \text{(Eq 16)}$$

$$\tau = \frac{4at}{\delta^2} \qquad \text{(Eq 17)}$$

$$\text{Bi} = \frac{h\delta}{2k} \qquad \text{(Eq 18)}$$

where Bi is the Biot number. It should be noted that the characteristic length is selected to be one-half of the penetration depth, $\delta/2$, within which the fraction of the power generation occurring is:

$$\frac{P_{\delta/2}}{P}=\frac{\int_0^{\delta/2}\dot{q}dx}{\int_0^\infty \dot{q}dx}=1-\frac{1}{e}=0.632 \qquad \text{(Eq 19)}$$

that is, 63% of the total power generation.

The temperature distribution as a function of these dimensionless parameters now becomes:

$$\begin{aligned}\theta(\xi, \tau) = &-\exp(-\xi)+\frac{1}{2}\exp(\tau-\xi)\operatorname{erfc}\left(\sqrt{\tau}-\frac{\xi}{2\sqrt{\tau}}\right)\\ &-\frac{1}{2}\left(\frac{\text{Bi}+1}{\text{Bi}-1}\right)\exp(\tau+\xi)\operatorname{erfc}\left(\sqrt{\tau}+\frac{\xi}{2\sqrt{\tau}}\right)\\ &+\left(\frac{\text{Bi}+1}{\text{Bi}}\right)\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right)+\frac{1}{\text{Bi}(\text{Bi}-1)}\\ &\cdot \exp(\text{Bi}^2\tau+\text{Bi}\xi)\operatorname{erfc}\left(\text{Bi}\sqrt{\tau}+\frac{\xi}{2\sqrt{\tau}}\right)\end{aligned} \qquad \text{(Eq 20)}$$

The temperature distribution, θ, in Eq 20 is plotted in Fig. 3 with respect to dimensionless distance, ξ, for selected values of dimensionless time, τ, and Bi = 0.005.

Consider a mild steel (0.23% C) sample to be heated by induction using 10^3 Hz frequency with a power density of P = 25 MW/m^2. The penetration depth, δ, is determined to be about 2 $\times 10^3$ m using Fig. 1 or Eq 8. Therefore, a surface temperature increase of 1000 °C, which corresponds to

$$\theta = \frac{2k}{\delta P}(T - T_\infty) = \frac{2\times 26}{2\times 10^{-3}\times 25\times 10^6}\times 1000 \approx 1$$

is obtained at $\tau \sim 3$; that is:

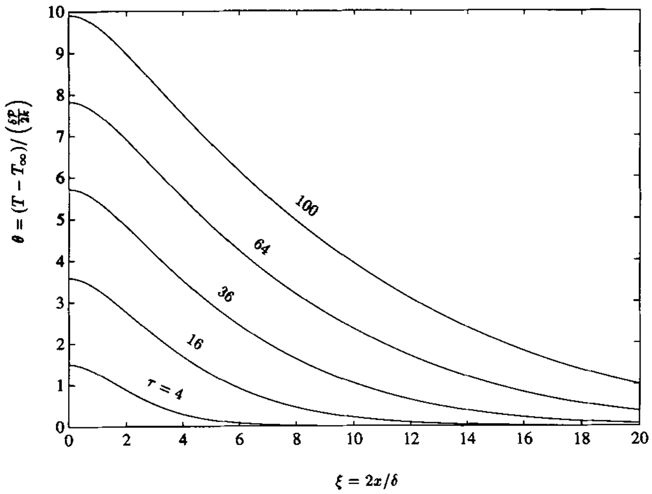

Fig. 3 Temperature rise during induction heating

$$t = \frac{\delta^2}{4a}\tau \approx \frac{(2 \times 10^{-3})^2}{4 \times 3.8648 \times 10^{-6}} \times 3 = 0.78 \text{ s}$$

The heat-affected zone (HAZ) extends up to $\xi \sim 4$; that is, $x \sim 2\delta = 4 \times 10^{-3}$ m = 4 mm.

Using a frequency of 10^5 Hz in this example, the penetration depth, δ, is found to be 2×10^{-4} m. In this case, a surface temperature increase of 1000 °C for the same power density used before corresponds to

$$\theta = \frac{2 \times 26}{2 \times 10^{-4} \times 25 \times 10^6} \times 1000 \approx 10$$

which gives $\tau \sim 100$; that is:

$$t \approx \frac{(2 \times 10^{-4})^2}{4 \times 3.8648 \times 10^{-6}} \times 100 = 0.26\ s$$

Here the HAZ extends up to $\xi \sim 20$; that is, $x \sim 10\delta = 2 \times 10^{-3}$ m = 2 mm. The HAZ can be further decreased by increasing the applied frequency.

4. Local Heat Flux

The dimensionless local heat flux can be defined as:

$$q'' = \frac{-k(dT/dx)}{P} = -\frac{\partial\theta}{\partial\varepsilon} \qquad \text{(Eq 21)}$$

Then, the local heat flux as a function of the nondimensional parameters (Eq 15 to 18) is:

$$q''(\xi, \tau) = -\exp(-\xi) + \frac{1}{2}\left(\frac{\text{Bi}+1}{\text{Bi}-1}\right)\exp(\tau+\xi)\,\text{erfc}\left(\sqrt{\tau} + \frac{\xi}{2\sqrt{\tau}}\right)$$

$$+\frac{1}{2}\exp(\tau-\xi)\,\text{erfc}\left(\sqrt{\tau} - \frac{\xi}{2\sqrt{\tau}}\right) - \frac{1}{(\text{Bi}-1)}$$

$$\cdot \exp(\text{Bi}^2\tau + \text{Bi}\xi)\,\text{erfc}\left(\text{Bi}\sqrt{\tau} + \frac{\xi}{2\sqrt{\tau}}\right) \qquad \text{(Eq 22)}$$

Note that the heat flux at the surface

$$q''(0, \tau) = -1 + \left(\frac{\text{Bi}}{\text{Bi}-1}\right)\exp(\tau)\,\text{erfc}(\sqrt{\tau})$$

$$-\frac{1}{(\text{Bi}-1)}\exp(\text{Bi}^2\tau)\,\text{erfc}(\text{Bi}\sqrt{\tau}) \qquad \text{(Eq 23)}$$

varies from 0 for $\tau = 0$ to -1 for $\tau \to \infty$. Therefore, the local heat flux on the surface is never zero for $t > 0$. This shows that the maximum temperature occurs inside the material, and the location of the maximum temperature is the value of ξ that makes Eq 22 zero.

5. Steady-State Solution

For $\tau \to \infty$, the temperature profile (Eq 20) gives:

$$\theta(\xi) = \frac{\text{Bi}+1}{\text{Bi}} - e^{-\xi} \qquad \text{(Eq 24)}$$

The steady-state surface temperature is therefore:

$$\theta(\xi = 0) = \frac{1}{\text{Bi}} \qquad \text{(Eq 25)}$$

while the maximum steady-state temperature in the material is:

$$\theta(\xi \to \infty) = 1 + \frac{1}{\text{Bi}} \qquad \text{(Eq 26)}$$

This is the maximum temperature inside the material that can be obtained during electromagnetic heating.

Steady-state local heat flux is obtained from Eq 22 by substituting $\tau \to \infty$:

$$q''(\xi) = -e^{-\xi} \qquad \text{(Eq 27)}$$

and the steady-state heat flux becomes:

$$q''(\xi = 0) = -1 \qquad \text{(Eq 28)}$$

as expected.

6. High-Frequency Solution

When the applied frequency, ω, is increased, the penetration depth, δ, decreases. In the limit when $\omega \to \infty$, no penetration occurs and all the energy is concentrated on the surface of the material, diffusing inward by conduction. However, in this case, the governing partial-differential equation and the boundary conditions become:

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} \quad \text{(Eq 29)}$$

$$T(x, 0) = T_\infty$$

$$-k\frac{\partial T}{\partial x}(0, t) = h[T_\infty - T(0, t)] + P$$

$$T(\infty, t) = \text{finite}$$

where P is the surface heat flux.

To obtain the solution of the earlier problem, let

$$\phi = T - T_\infty - \frac{P}{h}$$

Thus, the formulation is converted into:

$$\frac{\partial \phi}{\partial t} = a\frac{\partial^2 \phi}{\partial x^2} \quad \text{(Eq 30)}$$

$$\phi(x, 0) = -\frac{P}{h}$$

$$-k\frac{\partial \phi}{\partial x}(0, t) + h\phi(0, t) = 0$$

$$\phi(\infty, t) = \text{finite}$$

whose solution can be shown to be (Ref 5):

$$\phi(x, t) = -\frac{P}{h}\left[\text{erf}\left(\frac{x}{2\sqrt{at}}\right) + \exp\left(\frac{h}{k}x + \frac{h^2at}{k^2}\right)\text{erfc}\left(\frac{x}{2\sqrt{at}} + \frac{h}{k}\sqrt{at}\right)\right] \quad \text{(Eq 31)}$$

Therefore, the temperature rise in this case is determined to be:

$$\frac{T - T_\infty}{P/h} = \text{erfc}\left(\frac{x}{2\sqrt{at}}\right) - \exp\left(\frac{h}{k}x + \frac{h^2at}{k^2}\right)\text{erfc}\left(\frac{x}{2\sqrt{at}} + \frac{h}{k}\sqrt{at}\right) \quad \text{(Eq 32)}$$

It should be noted that the temperature distribution for long time as $t \to \infty$ is:

$$T - T_\infty \to \frac{P}{h}$$

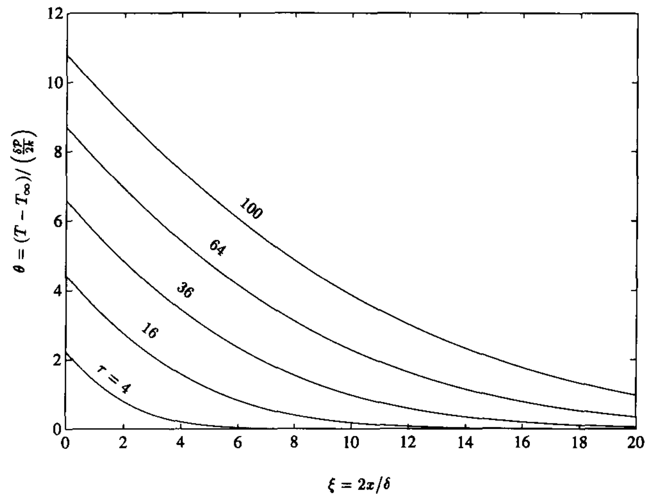

Fig. 4 Temperature rise during induction heating for $\omega \to \infty$

which is constant and inversely proportional to the heat-transfer coefficient, h.

The temperature rise for the high-frequency, $\omega \to \infty$ case is plotted as function of distance for selected rimes n Fig. 4 using the same dimensionless quantities and Bi number as in Fig. 3 for comparison. The temperature gradient in the vicinity of the surface is higher for high frequencies as expected. It should be noted that for the case of $\omega \to \infty$, the penetration depth is zero. Therefore, the value of δ used to generate Fig. 4 is a selected arbitrary characteristic thickness, which helps to nondimensionalize Eq 32.

The dimensionless local heat flux, on the other hand, is:

$$q''(x, t) = \frac{-k(dT/dx)}{P} = \exp\left(\frac{h}{k}x + \frac{h^2at}{k^2}\right)\text{erfc}\left(\frac{x}{2\sqrt{at}} + \frac{h}{k}\sqrt{at}\right) \quad \text{(Eq 33)}$$

It should be noted that the surface heat flux

$$q''(x = 0) = \exp\left(\frac{h^2at}{k^2}\right)\text{erfc}\left(\frac{h}{k}\sqrt{at}\right)$$

is an exponentially decaying function and varies from 1 at $t = 0$ to 0 at $t \to \infty$. Therefore, the maximum temperature occurs on the surface all the time.

7. Special Cases

7.1 *Constant Surface Temperature*

A specific constant surface temperature, T_∞, corresponds to $h \to \infty$ in the previous analysis. From Eq 20 the temperature profile in the semi-infinite domain in this case is determined to be:

$$\theta(\xi, \tau) = -\exp(-\xi) + \text{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right) - \frac{1}{2}\exp(\tau + \xi) \cdot \text{erfc}\left(\sqrt{\tau} + \frac{\xi}{2\sqrt{\tau}}\right) + \frac{1}{2}\exp(\tau - \xi)\,\text{erfc}\left(\sqrt{\tau} - \frac{\xi}{2\sqrt{\tau}}\right) \quad \text{(Eq 34)}$$

The steady-state temperature profile in this case is:

$$\theta(\xi) = 1 - e^{-\xi} \quad \text{(Eq 35)}$$

which means the maximum temperature is:

$$\theta(\xi \to \infty) = 1 \quad \text{(Eq 36)}$$

From Eq 22, the local heat flux variation in the material is found to be:

$$q''(\xi, \tau) = -\exp(-\xi) + \frac{1}{2}\exp(\tau + \xi)\,\text{erfc}\left(\sqrt{\tau} + \frac{\xi}{2\sqrt{\tau}}\right) + \frac{1}{2}\exp(\tau - \xi)\,\text{erfc}\left(\sqrt{\tau} - \frac{\xi}{2\sqrt{\tau}}\right) \quad \text{(Eq 37)}$$

Heat flux on the surface in this case is:

$$q''(0, \tau) = -1 + \exp(\tau)\,\text{erfc}(\sqrt{\tau}) \quad \text{(Eq 38)}$$

Steady-state local heat flux variation and the steady-state surface heat flux are the same as Eq 27 and 28, respectively.

7.2 *Adiabatic Surface*

In the case of adiabatic surface boundary condition, there is no steady-state solution. However, the solution of the problem in this case is given to be (Ref 5):

$$\theta(\xi, \tau) = -\exp(-\xi) + 2\sqrt{\tau}\left[\frac{1}{\sqrt{\pi}}\exp\left(-\frac{\xi^2}{4\tau}\right) - \frac{\xi}{2\sqrt{\tau}}\,\text{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right)\right] + \frac{1}{2}\exp(\tau + \xi)\,\text{erfc}\left(\sqrt{\tau} + \frac{\xi}{2\sqrt{\tau}}\right) + \frac{1}{2}\exp(\tau - \xi) \cdot \text{erfc}\left(\sqrt{\tau} - \frac{\xi}{2\sqrt{\tau}}\right) \quad \text{(Eq 39)}$$

For high frequency ($\omega \to \infty$, $\delta \to 0$):

$$T = T_\infty + 2\frac{P}{k}\left[\sqrt{\frac{at}{\pi}}\exp\left(-\frac{x^2}{4at}\right) - \frac{x}{2}\,\text{erfc}\left(\frac{x}{2\sqrt{at}}\right)\right] \quad \text{(Eq 40)}$$

Note that the temperature of the surface increases continuously with time, since:

$$T(x = 0) = T_\infty + \frac{2P}{k}\sqrt{\frac{at}{\pi}}$$

and this is the maximum temperature in the material during the induction heating process.

8. Conclusions

An analytical solution of the transient temperature distribution of an electrically conducting material exposed to electromagnetic radiation is obtained. The solution is carried out for the case where the surface is exposed to an ambient with convective heat losses. Due to the effect of convection on the surface, the maximum temperature is found to occur inside the material and not on the surface except for very high applied current frequencies. High-current frequency causes a decrease in the penetration depth and therefore yields large temperature gradients close to the surface of the material. The solutions for high heat-transfer coefficient and adiabatic surface boundary conditions are treated as special cases.

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